Journal of Functional Analysis **161**, 397–417 (1999)
Article ID jfan.1998.3356, available online at http://www.idealibrary.com on IDEAL®

Geometric Properties of Projections of Reproducing Kernels on z^*-Invariant Subspaces of H^2

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Communicated by D. Sarason

Received February 13, 1998; accepted July 31, 1998

Let θ be an inner function and let Λ be a Blaschke sequence in the unit disc. Denote by B the Blaschke product corresponding to Λ. Geometric properties of the family

$\theta B + H^{\infty}$ in L^{∞}/H^{∞} are established.

Key Words: Riesz bases; reproducing kernels; Schur's algorithm; extreme points.

0. INTRODUCTION

We consider the reproducing kernels $k(w, z)$ in the Hardy space H^2 on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $k(w, z) = 1/(1 - \bar{w}z)$. Let θ be an inner function. Denote by K_θ the subspace $H^2 \ominus \theta H^2$ of H^2 and by P_θ the operator of projection on this subspace.

Given a sequence $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$ of points in $\mathbb{D}$ satisfying the Blaschke condition, consider the corresponding Blaschke product $B = \prod_{i=0}^{\infty} b_{\lambda_i}$, where $b_\lambda(z) = (\bar{\lambda}/|\lambda|)(z - \lambda)/(1 - \bar{\lambda}z)$.

Put $\mathscr{K}_\Lambda = \{k(\lambda_i, \cdot)\}_{i=0}^{\infty}$. Nikolskiĭ proved (see [N, Lecture 6]) that the family $\mathscr{K}_\Lambda$ possesses the following property: it forms an unconditional basis in the closure of its linear hull if and only if it is uniformly minimal (for definitions see Section 1). A necessary and sufficient condition for $\mathscr{K}_\Lambda$ to be an unconditional basis (or a uniformly minimal family) is the Carleson condition on Λ:

$$\inf_k \left| \frac{B}{b_{\lambda_k}}(\lambda_k) \right| > 0. \tag{C}$$

Next, consider the family $\mathscr{K}_{\Lambda,\theta}$ of the projections of elements of $\mathscr{K}_\Lambda$ on K_θ, $\mathscr{K}_{\Lambda,\theta} = \{k_\theta(\lambda_i, \cdot)\}_{i=0}^{\infty}$, $k_\theta(\lambda, \cdot) = P_\theta k(\lambda, \cdot)$, $k_\theta(\lambda, z) = (1 - \overline{\theta(\lambda)}\,\theta(z))/(1 - \bar{\lambda}z)$. Nikolskiĭ proved that the family $\mathscr{K}_{\Lambda,\theta}$ forms an unconditional

0022-1236/99 $30.00

basis in the closure of its linear hull if the set Λ satisfies the Carleson condition (C) and, additionally,

$$\operatorname{dist}_{L^\infty}(\theta, BH^\infty) < 1. \tag{0.1}$$

In our earlier paper [B], we noticed that there is some relationship between condition (0.1) and the behavior of the series of Schur–Nevanlinna coefficients. Here we obtain new results confirming this point of view.

Let us give necessary definitions. Fix an inner function θ and a Blaschke product $B = \prod_{i=0}^{\infty} b_{\lambda_i}$. We construct the following sequences of Schur–Nevanlinna coefficients $\{\gamma_n\}_{n=0}^{\infty}$ and functions $\{\theta_n\}_{n=0}^{\infty}$:

$$\left.\begin{array}{ll} \theta_0(z) = \theta(z), & \gamma_0 = \theta_0(\lambda_0), \\ \theta_1(z) = \dfrac{\theta_0(z) - \gamma_0}{1 - \overline{\gamma_0}\theta_0(z)} \dfrac{1 - \overline{\lambda_0} z}{z - \lambda_0}, & \gamma_1 = \theta_1(\lambda_1), \\ \dots & \\ \theta_n(z) = \dfrac{\theta_{n-1}(z) - \gamma_{n-1}}{1 - \overline{\gamma_{n-1}}\theta_{n-1}(z)} \dfrac{1 - \overline{\lambda_{n-1}} z}{z - \lambda_{n-1}}, & \gamma_n = \theta_n(\lambda_n), \\ \dots & \end{array}\right\} \tag{0.2}$$

In this paper we express the distance

$$\operatorname{dist}_{H^2}(k_\theta(\lambda_k, \cdot), \operatorname{span}(k_\theta(\lambda_i, \cdot) : 0 \leqslant i \leqslant n, i \neq k)) \tag{0.3}$$

in terms of the Schur–Nevanlinna coefficients and functions and obtain criteria of minimality and uniform minimality for the family $\mathscr{K}_{\Lambda, \theta}$. (We use the notation $\operatorname{span}(E)$ for the closed linear hull of E.) In particular, we see that uniform minimality of $\mathscr{K}_{\Lambda, \theta}$ implies the Carleson condition (C). This result was proved in [HNP; N, Lecture 8] only under the additional assumption that $\sup_{\lambda \in \Lambda} |\theta(\lambda)| < 1$.

Furthermore, we obtain the formula for the above distance

$$\begin{aligned} &\operatorname{dist}^2_{H^2}\left(\frac{k_\theta(\lambda_k, \cdot)}{\|k_\theta(\lambda_k, \cdot)\|}, \operatorname{span}(k_\theta(\lambda_i, \cdot) : 0 \leqslant i \leqslant n, i \neq k)\right) \\ &\quad = \left|\frac{B}{b_{\lambda_k}}(\lambda_k)\right|^2 \frac{1}{1 - |\theta(\lambda_k)|^2} \max_{f \in (\theta + B_n H^\infty) \cap \mathscr{B}} \\ &\quad\quad \times \exp \int_0^{2\pi} \log(1 - |f(e^{i\varphi})|^2)\, P(\varphi; \lambda_k)\, dm(\varphi), \end{aligned} \tag{0.4}$$

where $P(\varphi; \mu) = (1 - |\mu|^2)/|e^{i\varphi} - \mu|^2$, $|\mu| < 1$, is the Poisson kernel, $\mathscr{B} = \{f \in H^\infty : \|f\|_\infty \leqslant 1\}$, and m denotes the normalized Lebesgue measure.

A theorem of de Leeuw and Rudin (see [H, Chap. 9]) states that the condition

$$\exp \int_0^{2\pi} \log(1 - |f(e^{i\varphi})|^2)\, P(\varphi; \lambda)\, dm(\varphi) > 0,$$

for some (every) $\lambda \in \mathbb{D}$, is equivalent to the property that f is a non-extreme point of $\mathscr{B}$. Furthermore, a result of P. Koosis [K] shows that given two inner functions θ, B, the class $(\theta + BH^\infty) \cap \mathscr{B}$ contains only extreme elements of $\mathscr{B}$ if and only if $\theta\bar{B} + H^\infty$ is an extreme point of the closed unit ball of L^∞/H^∞; in this situation we have $(\theta + BH^\infty) \cap \mathscr{B} = \{\theta\}$.

Summing up, we obtain

THEOREM 4.7. *Let θ be an inner function and let B be the Blaschke product constructed by a sequence Λ of distinct points. Then the following assertions are equivalent*:

(a) *The family $\mathscr{K}_{\Lambda, \theta}$ is not minimal.*

(b) *For some (every) $\mu \in \Lambda$ the family $\mathscr{K}_{\Lambda \setminus \{\mu\}, \theta}$ is complete in K_θ.*

(c) *For some (every) $\mu \in \Lambda$, $P_\theta(B/(z-\mu)) \in \operatorname{clos} P_\theta(K_{B/b_\mu})$.*

(d) *For some (every) $\mu \in \Lambda$, $P_\theta(B/(z-\mu)\, H^2) \subset \operatorname{clos} P_\theta(K_{B/b_\mu})$.*

(e) *The class $\theta\bar{B} + H^\infty$ is an extreme point of the closed unit ball of L^∞/H^∞.*

(f) $(\theta + BH^\infty) \cap \mathscr{B} = \{\theta\}$.

In view of Propositions D, D′ of Section 1 it seems to be interesting to compare the above theorem with the following two facts, see Sarason [S, Chap. 9] and Nikolskiĭ [N, Lecture 8]: (a) if $\operatorname{dist}_{L^\infty}(\theta, BH^\infty) < \operatorname{dist}_{L^\infty}(\theta, zBH^\infty)$, then the class $\theta + zBH^\infty$ contains a unique function of minimal norm; (b) if $\operatorname{dist}_{L^\infty}(\theta, BH^\infty) < 1$, then $\operatorname{dist}_{L^\infty}(\theta, zBH^\infty) < 1 \Leftrightarrow \operatorname{dist}_{L^\infty}(B, \theta H^\infty) = 1$.

Generally speaking, to be minimal is a condition weaker than to be a basis, as well as the condition that the class $\theta + BH^\infty$ contains a non-extreme point of $\mathscr{B}$ is weaker than condition (0.1). However, in one particular case all these conditions coincide.

COROLLARY 4.8. *Let Λ satisfy the Carleson condition* (C) *and let* $\lim_{n \to \infty} \theta(\lambda_n) = 0$. *Then the following assertions are equivalent*:

(a) *The family $\mathscr{K}_{\Lambda, \theta}$ is minimal.*

(b) *The family $\mathscr{K}_{\Lambda, \theta}$ is an unconditional basis in the closure of its linear hull.*

(c) *The class* $\theta + BH^\infty$ *contains a non-extreme point of* $\mathscr{B}$.

(d) $\operatorname{dist}_{L^\infty}(\theta, BH^\infty) < 1$.

In Section 1 we give necessary facts and definitions. In Section 2 we compute the Gram determinants of families $\mathscr{K}_{\Lambda,\theta}$. In Section 3 we express the distance (0.3) in terms of the Schur–Nevanlinna coefficients and functions. Finally, in Section 4 we obtain formula (0.4) and prove our main results, i.e., Theorem 4.7 and Corollary 4.8.

The author is thankful to Konstantin Dyakonov and Nicolai Nikolskiĭ for helpful discussions.

1. GENERAL INFORMATION

A family $\{f_n\}_{n=1}^\infty$ of elements in a Banach space is said to be *minimal* if for every n the element f_n does not belong to the closure of the linear hull of $\{f_k, k \neq n\}$, and *uniformly minimal* if

$$\inf_n \operatorname{dist}(f_n/\|f_n\|, \operatorname{span}(\{f_k : k \neq n\})) > 0.$$

Given a family of vectors $\mathscr{E}$ we denote by $\Gamma(\mathscr{E})$ the Gram matrix of $\mathscr{E}$.

We use the following result from [AG] to determine whether a family is minimal (uniformly minimal).

Proposition A. *Let* $\mathscr{E} = \{e_i\}_{i=1}^k$, $k \geqslant 1$ *be a linearly independent family of elements of a Hilbert space* H. *Put* $\mathscr{E}_j = \mathscr{E} \setminus \{e_j\}$. *Denote the Gram matrix of the family* $\mathscr{E}$ *by* $\Gamma(\mathscr{E})$. *Then*

$$\operatorname{dist}^2_H(e_j, \operatorname{span}(\mathscr{E}_j)) = \frac{\det \Gamma(\mathscr{E})}{\det \Gamma(\mathscr{E}_j)}.$$

We need also the following result connecting Gram matrices with determinants of operators; see [GL].

Proposition B. *Let* $\mathscr{A}, \mathscr{D}$ *be operators on a finite-dimensional Hilbert space* H. *Let* F *be an orthonormal basis in* H. *Denote by* A_F *the matrix of* $\mathscr{A}$ *in this basis. Then*

(a) *The determinant* $\det A_F$ *does not depend on the choice of the basis* F. *We denote it by* $\det \mathscr{A}$.

(b) $|\det \mathscr{A}|^2 = \det \Gamma(\mathscr{A}F)$

(c) $\det \Gamma((\mathscr{D}\mathscr{A})F) = \det \Gamma(\mathscr{D}F) \det \Gamma(\mathscr{A}F) = \det \Gamma(\mathscr{D}F)\, |\det \mathscr{A}|^2$.

A family $\{f_n\}_{n=1}^{\infty}$ of vectors in a Hilbert space H is called *an unconditional basis* if it is complete in H and for every finite complex sequence $\{a_n\}$, $\|\sum a_n f_n\|^2$ and $\sum |a_n| \, \|f_n\|^2$ are comparable.

We use some results from [HNP; N, Lectures 6–8].

PROPOSITION C. *The following assertions are equivalent*:

(a) *The family $\mathcal{K}_\Lambda$ forms an unconditional basis in the closure of its linear hull.*

(b) *The family $\mathcal{K}_\Lambda$ is uniformly minimal.*

(c) *The set Λ satisfies the Carleson condition* (C).

PROPOSITION D. *Let the set Λ satisfy the Carleson condition* (C) *and suppose that* $\sup_{\lambda \in \Lambda} |\theta(\lambda)| < 1$. *Then the following assertions are equivalent*:

(a) *Condition* (0.1) *is fulfilled.*

(b) *The family $\mathcal{K}_{\Lambda,\theta}$ forms an unconditional basis in the closure of its linear hull.*

PROPOSITION D′. *Let the set Λ satisfy the Carleson condition* (C) *and suppose that* $\sup_{\lambda \in \Lambda} |\theta(\lambda)| < 1$. *Then the following assertions are equivalent*:

(a) $\operatorname{dist}_{L^\infty}(\theta, BH^\infty) < 1$ *and* $\operatorname{dist}_{L^\infty}(B, \theta H^\infty) < 1$.

(b) *The family $\mathcal{K}_{\Lambda,\theta}$ forms an unconditional basis in K_θ.*

PROPOSITION E. *The family $\mathcal{K}_{\Lambda \setminus \{\mu\},\theta}$ is complete in K_θ for some $\mu \in \Lambda$ if and only if it is complete for all $\mu \in \Lambda$.*

To prove this last assertion we argue as in [N, p. 211]: if $f \in K_\theta$, $f(\mu) = 0$, then $g = f(1 - b_\mu(\mu_1)/b_\mu) \in K_\theta$, $g(\mu_1) = 0$.

We use the Malmquist–Walsh basis in the space K_B. Given a Blaschke product $B(z) = \prod_{i=0}^{\infty} b_{\lambda_i}(z)$, put $B_{-1}(z) \equiv 1$, $B_k(z) = \prod_{i=0}^{k} b_{\lambda_i}(z)$, $k = 0, 1, \ldots$. Then $L = \{l_i\}_{i=0}^{\infty}$, where

$$l_i(z) = B_{i-1}(z) \frac{(1 - |\lambda_i|^2)^{1/2}}{1 - \bar{\lambda}_i z}, \qquad i = 0, 1, \ldots, \tag{1.1}$$

is an orthonormal basis of K_B (see [W]).

We need the following important fact, see [N, Lectures 2, 3, 5] for complete formulations and references: L is a basis of low-triangle representation for the model operator $T_B : K_B \to K_B$,

$$T_B g = P_B M_z g,$$

where M_z is the operator of multiplication by z in H^2. Moreover, it is easily seen that $L_n=\{l_i\}_{i=0}^n$ is a basis of low-triangle representation for the operator T_{B_n}, $n=0, 1, \ldots$.

We need also some properties of classes $\theta+BH^\infty$ in H^∞, $\theta\overline{B}+H^\infty$ in L^∞/H^∞. The first of the following results was obtained by Koosis [K] and the second by Stray [Str].

Proposition F. *Let $g\in L^\infty$, $\|g\|_\infty=1$. The class $g+H^\infty$ is an extreme point of the closed unit ball in L^∞/H^∞ if and only if $\|g+h\|>1$ for every non-zero $h\in H^\infty$.*

Proposition G. *Let the set Λ satisfy the condition* (C) *and let $\theta(\lambda_n)\to 0$ as $n\to\infty$. Then the class $\theta+BH^\infty$ contains a unique element of minimal norm which is a complex constant times a Blaschke product.*

2. TECHNICAL PRELIMINARIES

We consider an inner function θ and a Blaschke product $B(z)=\prod_{i=0}^\infty b_{\lambda_i}(z)$. Put $\Lambda=\{\lambda_i\}_{i=0}^\infty$, $B_n=\prod_{i=0}^n b_{\lambda_i}(z)$.

Our first aim is to compute the determinant of the Gram matrix $\Gamma_n=\Gamma(P_\theta L_n)$, where $L_n=\{l_i\}_{i=0}^n$ is the Malmquist–Walsh basis of K_{B_n}.

Note that we have $(P_\theta l_i, P_\theta l_j)=(l_i, P_\theta l_j)$, because P_θ is a projection operator. Thus, Γ_n^T coincides with the matrix of the restriction of the operator P_θ on span$(\{l_i\}_{i=0}^n)$ and, consequently, its determinant does not vary under transpositions of the sequence Λ in the definition (1.1).

Proposition 2.1. *Let the functions $\{\theta_i\}_{i=0}^n$ and the numbers $\{\gamma_i\}_{i=0}^n$ be defined by (θ, Λ) using the Schur–Nevanlinna process* (0.2). *Then*

$$\det \Gamma_n=\frac{\prod_{i=0}^{n-1}\prod_{l=i+1}^n |1-\overline{\gamma_i}\,\theta_i(\lambda_l)|^2}{\prod_{i=0}^n (1-|\gamma_i|^2)^{n-i-1}}. \tag{2.1}$$

As a corollary we obtain the formula by Schur [Sch]

$$\det \Gamma_n=\prod_{l=0}^n (1-|\gamma_l|^2)^{n+1-l},$$

corresponding to the case $\lambda_0=\lambda_1=\cdots=\lambda_n=0$.

To prove Proposition 2.1 we use a functional calculus described in [N, Lecture 3]. Given $\varphi \in L^\infty$, denote by M_φ the operator of multiplication by φ. Let V be an inner function. We define $\varphi(T_V) = P_V M_\varphi \mid K_V$, where T_V is the model operator defined in Section 1. It is proved in [N, Lecture 3] that

$$[\varphi(T_V)]^* = P_+ M_{\bar{\varphi}} \mid K_V. \tag{2.2}$$

We consider the operator $\mathscr{A}_n = \theta(T_{B_n})$. Denote by A_n its matrix in the basis L_n. Since the matrix of T_{B_n} is lower-triangle (see the properties of L_n in Section 1) and $\theta \in H^\infty$, the matrix A_n is also lower-triangle.

Denote by $\mathscr{I}$ the identity operator and by I_k the identity matrix of order k.

LEMMA 2.2. *$P_{B_n} P_\theta \mid K_{B_n} = \mathscr{I} - \mathscr{A}_n \mathscr{A}_n^*$, and as a consequence, $\Gamma_n = I_{n+1} - A_n A_n^*$.*

Proof of Lemma 2.2. By the Lemma on Projection [N, Lecture 2], $P_\theta = \mathscr{I} - M_\theta P_+ \overline{M_\theta}$. Therefore,

$$\begin{aligned} P_{B_n} P_\theta \mid K_{B_n} &= P_{B_n} (\mathscr{I} - M_\theta P_+ \overline{M_\theta}) \mid K_{B_n} \\ &= \mathscr{I} - P_{B_n} M_\theta P_+ M_{\bar{\theta}} \mid K_{B_n} \\ &= \mathscr{I} - \mathscr{A}_n \mathscr{A}_n^*. \quad \blacksquare \end{aligned}$$

We prove Proposition 2.1 by induction; we reduce the dimension by passing from the space K_{B_n} to the spaces $K_{B_{k,n}}$, $k = 0, ..., n$, $B_{k,n} = \prod_{i=k}^n b_{\lambda_i}$. In this process we use the orthonormal bases $L_{k,n} = \{l_{k,i}\}_{i=0}^{n-k}$ of $K_{B_{k,n}}$,

$$l_{k,i}(z) = B_{k,k+i-1}(z) \frac{(1 - |\lambda_{k+i}|^2)^{1/2}}{1 - \overline{\lambda_{k+i}} z}, \qquad 0 \leqslant i \leqslant n-k; \quad B_{k,k-1} \equiv 1.$$

The basis $L_{0,n}$ is just the basis L_n defined by (1.1).

LEMMA 2.3. *Let $\varphi \in H^\infty$ and let k, n be natural numbers, $0 \leqslant k \leqslant n$. Denote by $F_{k,n}$ the matrix of the operator $\mathscr{F}_{k,n} = \varphi(T_{B_{k,n}})$ in the basis $L_{k,n}$. Then for $0 \leqslant k \leqslant n$*

$$(F_{k,n})_{i+1,j+1} = (F_{k+1,n})_{i,j}, \qquad 0 \leqslant i, j \leqslant n-k-1.$$

Proof of Lemma 2.3. We have $M_\varphi \mid K_{B_n} - \varphi(T_{B_{k,n}}) = M_{B_{k,n}} P_+ M_{\overline{B_{k,n}}} M_\varphi \mid K_{B_n}$. Therefore,

$$
\begin{aligned}
&((M_\varphi - \mathscr{F}_{k,n})\, l_{k,\,j+1},\, l_{k,\,i+1})\\
&\quad = \left(B_{k,n}(z)\, P_+ \overline{B_{k,n}(z)}\, \varphi(z) B_{k,\,k+j}(z)\, \frac{(1-|\lambda_{k+j+1}|^2)^{1/2}}{1-\overline{\lambda_{k+j+1}}z},\right.\\
&\qquad\qquad \left. B_{k,\,k+i}(z)\, \frac{(1-|\lambda_{k+i+1}|^2)^{1/2}}{1-\overline{\lambda_{k+i+1}}z}\right)\\
&\quad = \left(B_{k+1,\,n}(z)\, P_+ \overline{B_{k+1,\,n}(z)}\, \varphi(z)\, B_{k+1,\,k+j}(z)\, \frac{(1-|\lambda_{k+j+1}|^2)^{1/2}}{1-\overline{\lambda_{k+j+1}}z},\right.\\
&\qquad\qquad \left. B_{k+1,\,k+i}(z)\, \frac{(1-|\lambda_{k+i+1}|^2)^{1/2}}{1-\overline{\lambda_{k+i+1}}z}\right)\\
&\quad = ((M_\varphi - \mathscr{F}_{k+1,\,n})\, l_{k+1,\,j},\, l_{k+1,\,i}).
\end{aligned}
$$

Analogously, $(M_\varphi l_{k,\,j+1}, l_{k,\,i+1}) = (M_\varphi l_{k+1,\,j}, l_{k+1,\,i})$. This proves the lemma. ∎

LEMMA 2.4. *Put* $\mathscr{D}_{k,n} = b_{\lambda_k}(T_{B_{k,n}})$ *and let* $D_{k,n}$ *be the matrix of this operator in the basis* $L_{k,n}$. *Then*

$$
D_{k,n}(D_{k,n})^* = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & I_{n-k+1} & \\ 0 & & & \end{pmatrix}.
$$

Proof of Lemma 2.4. By (2.2), $(\mathscr{D}_{k,n})^* = P_+ M_{\overline{b_{\lambda_k}}} \mid K_{B_{k,n}}$. Since $\overline{b_{\lambda_k}}\, l_{k,\,i} = l_{k+1,\,i-1}$ for $i \geqslant 1$ and $\overline{b_{\lambda_k}}\, l_{k,\,0} \in L^2 \ominus H^2$, we have

$$
\begin{aligned}
(D_{k,n}(D_{k,n})^*)_{i,\,j} &= (\mathscr{D}_{k,n}(\mathscr{D}_{k,n})^*\, l_{k,\,j},\, l_{k,\,i}) = ((\mathscr{D}_{k,n})^*\, l_{k,\,j},\, (\mathscr{D}_{k,n})^*\, l_{k,\,i})\\
&= \begin{cases} (P_+ l_{k+1,\,j-1},\, P_+ l_{k+1,\,i-1}), & 1 \leqslant i,\, j \leqslant n-k,\\ 0, & ij = 0. \end{cases}\\
&= \begin{cases} \delta_{i,\,j}, & 1 \leqslant i,\, j \leqslant n-k,\\ 0, & ij = 0. \end{cases} \quad ∎
\end{aligned}
$$

Now we consider the Schur–Nevanlinna process (0.2) for the pair $(\theta; \Lambda)$ and obtain the sequences $\{\theta_l(z)\}_{l=0}^n$ and $\{\gamma_l\}_{l=0}^n$. Put $\mathscr{A}^l_{k,n} = \theta_l(T_{B_{k,n}}) = P_{K_{B_{k,n}}} M_{\theta_l} \mid K_{B_{k,n}}$, $0 \leqslant k, l \leqslant n$. We denote $\mathscr{A}^k_{k,n}$ by $\mathscr{A}_{k,n}$. Let $A^l_{k,n}$ be the matrix of $\mathscr{A}^l_{k,n}$ in the basis $L_{k,n}$ (respectively, $A_{k,n}$ denotes the matrix of $\mathscr{A}_{k,n}$ in this basis). Then $A_n = A_{0,n}$.

LEMMA 2.5. *Put* $\widetilde{\theta}_k(z)=(\theta_k(z)-\gamma_k)/(1-\overline{\gamma_k}\theta_k(z))$, $\widetilde{\mathscr{A}}_{k,n}=\widetilde{\theta}_k(T_{B_{k,n}})$ *and let* $\widetilde{A}_{k,n}$ *be the matrix of* $\widetilde{\mathscr{A}}_{k,n}$ *in the basis* $L_{k,n}$. *Then*

$$\widetilde{A}_{k,n}(\widetilde{A}_{k,n})^*=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_{k+1,n}(A_{k+1,n})^* & \\ 0 & & & \end{pmatrix}. \tag{2.3}$$

Proof of Lemma 2.5. By Lemma 2.3 (with $\varphi=\theta_{k+1}$) and the fact that $A_{k,n}^{k+1}$ is lower-triangle, we have

$$A_{k,n}^{k+1}=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ * & & & \\ \vdots & & A_{k+1,n} & \\ * & & & \end{pmatrix}. \tag{2.4}$$

Since $\widetilde{\theta}_k(z)=\theta_{k+1}(z)\,b_{\lambda_k}(z)$, the multiplicativity of the functional calculus implies that $\widetilde{\mathscr{A}}_{k,n}=\mathscr{A}_{k,n}^{k+1}b_{\lambda_k}(T_{B_{k,n}})$, and consequently,

$$\widetilde{\mathscr{A}}_{k,n}(\widetilde{\mathscr{A}}_{k,n})^*=\mathscr{A}_{k,n}^{k+1}b_{\lambda_k}(T_{B_{k,n}})(b_{\lambda_k}(T_{B_{k,n}}))^*\,(\mathscr{A}_{k,n}^{k+1})^*,$$
$$\widetilde{A}_{k,n}(\widetilde{A}_{k,n})^*=A_{k,n}^{k+1}\,D_{k,n}(D_{k,n})^*\,(A_{k,n}^{k+1})^*,$$

where $D_{k,n}$ are defined in Lemma 2.4. This lemma and the formula (2.4) complete the proof. ▮

LEMMA 2.6. *Let* $r\in\mathbb{N}$, $\lambda\in\mathbb{C}$ *and let* B *be an* $r\times r$*-matrix such that* $I_r-\bar{\lambda}B$ *is invertible. Put*

$$\widetilde{B}=(B-\lambda I_r)(I_r-\bar{\lambda}B)^{-1}. \tag{2.5}$$

Then

$$I_r-\widetilde{B}\widetilde{B}^*=(1-|\lambda|^2)(I_r-\bar{\lambda}B)^{-1}\,(I_r-BB^*)(I_r-\lambda B^*)^{-1}. \tag{2.6}$$

A similar formula is given in [BC, Chap. 4]. Note that (2.6) is a matrix analog of the well-known equality

$$1-\left|\frac{a-b}{1-\bar{b}a}\right|^2=\frac{(1-|a|^2)(1-|b|^2)}{|1-\bar{b}a|^2}, \tag{2.7}$$

Proof of Lemma 2.6. We are to prove that

$$(I_r-\bar{\lambda}B)(I_r-\widetilde{B}\widetilde{B}^*)(I_r-\lambda B^*)=(1-|\lambda|^2)(I_r-BB^*). \tag{2.8}$$

Substituting (2.5) in the left-hand side of (2.8) and taking into account the fact that functions of the same operator commute, we obtain that

$$\begin{aligned}
&(I_r-\bar{\lambda}B)(I_r-\widetilde{B}\widetilde{B}^*)(I_r-\lambda B^*)\\
&\quad=(I_r-\bar{\lambda}B)[I_r-(B-\lambda I_r)(I_r-\bar{\lambda}B)^{-1}\\
&\qquad\times(I_r-\lambda B^*)^{-1}(B^*-\bar{\lambda}I_r)](I_r-\lambda B^*)\\
&\quad=[I_r-\bar{\lambda}B-(I_r-\bar{\lambda}B)(I_r-\bar{\lambda}B)^{-1}(B-\lambda I_r)\\
&\qquad\times(I_r-\lambda B^*)^{-1}(B^*-\bar{\lambda}I_r)](I_r-\lambda B^*)\\
&\quad=[I_r-\bar{\lambda}B-(B-\lambda I_r)(B^*-\bar{\lambda}I_r)(I_r-\lambda B^*)^{-1}](I_r-\lambda B^*)\\
&\quad=(I_r-\bar{\lambda}B)(I_r-\lambda B^*)-(B-\lambda I_r)(B^*-\bar{\lambda}I_r)\\
&\quad=I_r-\bar{\lambda}B-\lambda B^*+|\lambda|^2BB^*-BB^*+\bar{\lambda}B+\lambda B^*-|\lambda|^2I_r\\
&\quad=(1-|\lambda|^2)(I_r-BB^*). \quad \blacksquare
\end{aligned}$$

LEMMA 2.7.

$$\begin{aligned}
I_{n-k+1}-\widetilde{A}_{k,n}\widetilde{A}^*_{k,n}=(1-|\gamma_k|^2)&(I_{n-k+1}-\overline{\gamma_k}A_{k,n})^{-1}\\
&\times(I_{n-k+1}-A_{k,n}(A_{k,n})^*)\\
&\times(I_{n-k+1}-\gamma_k(A_{k,n})^*)^{-1}. \qquad (2.9)
\end{aligned}$$

Proof of Lemma 2.7. Since $\mathscr{A}_{k,n}=\theta_k(T_{B_{k,n}})$ and $\widetilde{\mathscr{A}}_{k,n}=\widetilde{\theta}_k(T_{B_{k,n}})$, the multiplicativity of the functional calculus gives us that

$$\widetilde{\mathscr{A}}_{k,n}=(\mathscr{A}_{k,n}-\gamma_k I)(I-\overline{\gamma_k}\mathscr{A}_{k,n})^{-1}.$$

Correspondingly, for the matrices we have

$$\widetilde{A}_{k,n}=(A_{k,n}-\gamma_k I)(I-\overline{\gamma_k}A_{k,n})^{-1},$$

and (2.9) follows immediately from (2.6). $\blacksquare$

LEMMA 2.8.

$$\det(I_{n-k+1}-\overline{\gamma_k}A_{k,n})=\prod_{i=k}^{n}(1-\overline{\gamma_k}\theta_k(\lambda_i)).$$

This lemma is a simple corollary of the fact that the basis $\mathrm{L}_{k,n}$ is the basis of triangle representation for $\mathscr{A}_{k,n}$ and the eigenvalues of $\mathscr{A}_{k,n}$ are the points $\theta_k(\lambda_i)$, $i=k, k+1, ..., n$; see [N, Lecture 3].

Proof of Proposition 2.1. We consider the finite sequence of matrices $\{A_{k,n}\}_{k=0}^{n}$. Recall that $A_{k,n}$, $k \geqslant 0$ is the $(n-k+1)\times(n-k+1)$-matrix of the operator $\theta_k(T_{B_{k,n}})$, so that $A_n = A_{0,n}$. Formulas (2.3) and (2.9) imply that

$$\begin{aligned}\det(I_{n-k} - A_{k+1,n}(A_{k+1,n})^*) &= \det(I_{n-k+1} - \tilde{A}_{k,n}(\tilde{A}_{k,n})^*)\\ &= (1-|\gamma_k|^2)^{n-k+1}\det(I_{n-k+1} - \overline{\gamma_k}A_{k,n})^{-1}\\ &\quad\times \det(I_{n-k+1} - A_{k,n}(A_{k,n})^*)\\ &\quad\times \det(I_{n-k+1} - \gamma_k(A_{k,n})^*)^{-1}.\end{aligned}$$

Using Lemmas 2.7, 2.8 we get the recurrent equality

$$\begin{aligned}\frac{\det(I_{n-k+1} - A_{k,n}(A_{k,n})^*)}{\det(I_{n-k} - A_{k+1,n}(A_{k+1,n})^*)} &= \frac{\prod_{l=k}^{n}|1-\overline{\gamma_k}\theta_k(\lambda_l)|^2}{(1-|\gamma_k|^2)^{n-k+1}}\\ &= (1-|\gamma_k|^2)^2\frac{\prod_{l=k+1}^{n}|1-\overline{\gamma_k}\theta_k(\lambda_l)|^2}{(1-|\gamma_k|^2)^{n-k+1}}\\ &= \frac{\prod_{l=k+1}^{n}|1-\overline{\gamma_k}\theta_k(\lambda_l)|^2}{(1-|\gamma_k|^2)^{n-k-1}}.\end{aligned}\tag{2.10}$$

Let us compute $\det(I_1 - A_{n,n}(A_{n,n})^*)$:

$$\begin{aligned}(A_{n,n}l_{n,0}, l_{n,0}) &= \left(b_{\lambda_n}P_{-}\overline{b_{\lambda_n}}\theta_n\frac{(1-|\lambda_n|^2)^{1/2}}{1-\overline{\lambda_n}z}, \frac{(1-|\lambda_n|^2)^{1/2}}{1-\overline{\lambda_n}z}\right)\\ &= \left((1-|\lambda_n|^2)\frac{z-\lambda_n}{1-\overline{\lambda_n}z}\frac{\theta_n(\lambda_n)}{z-\lambda_n}, \frac{1}{1-\overline{\lambda_n}z}\right)\\ &= \theta_n(\lambda_n).\end{aligned}$$

Hence,

$$\det(I_1 - A_{n,n}(A_{n,n})^*) = 1 - |\theta_n(\lambda_n)|^2 = 1 - |\gamma_n|^2.\tag{2.11}$$

Equalities (2.10) and (2.11) together with Lemma 2.2 prove the proposition. ∎

Let us return to the family $\mathscr{K}_{\Lambda,\theta} = \{k_\theta(\lambda_i,\cdot)\}_{\lambda_i\in\Lambda}$ of projections of reproducing kernels.

COROLLARY 2.9. *Let $\lambda_0, \lambda_1, ..., \lambda_n$ be a sequence of points of $\mathbb{D}$. Then*

$$\det \Gamma(\{k_\theta(\lambda_i, \cdot)\}_{i=0}^n) = \prod_{i=0}^n \frac{|B_{i-1}(\lambda_i)|^2}{1-|\lambda_i|^2} \frac{\prod_{i=0}^{n-1} \prod_{l=i+1}^n |1-\overline{\gamma}_i \theta_i(\lambda_l)|^2}{\prod_{i=0}^n (1-|\gamma_i|^2)^{n-i-1}}. \tag{2.12}$$

Proof. Denote by Q the linear operator acting on K_{B_n} and mapping l_i into $k(\lambda_i, \cdot)$, $i=0, 1, ..., n$. By Proposition B (with $\mathscr{A} = P_\theta \mid K_{B_n}$, $\mathscr{B} = Q$) we have

$$\det \Gamma(\{k_\theta(\lambda_i, \cdot)\}_{i=0}^n) = \det \Gamma((P_\theta \mid K_B)\, QL_n) = \det \Gamma(P_\theta L_n)\, |\det Q|^2. \tag{2.13}$$

Since $(k_{\lambda_i}, l_j) = 0$ for $i < j$, the matrix of Q in the basis L_n is low-triangle and we obtain

$$\det Q = \prod_{i=0}^n |(k(\lambda_i, \cdot), l_i)| = \prod_{i=0}^n \frac{|B_{i-1}(\lambda_i)|}{\sqrt{1-|\lambda_i|^2}}. \tag{2.14}$$

Substituting (2.14) into (2.13) and applying Proposition 2.1 we obtain (2.12). ∎

3. DISTANCE FORMULAS FOR PROJECTIONS OF REPRODUCING KERNELS

Recall that $\{l_i\}_{i=0}^\infty$ is the Malmquist–Walsh basis of K_B defined in (1.1). Given functions $\{\theta_i\}_{i=1}^n$ put

$$\Pi_n(z) = \prod_{i=1}^n \frac{1-|\theta_i(z)|^2}{1-|\theta_i(z)|^2\, |b_{\lambda_{i-1}}(z)|^2}, \qquad 1 \leqslant n \leqslant \infty.$$

PROPOSITION 3.1.

$$\operatorname{dist}^2_{H^2}(P_\theta l_n, \operatorname{span}(P_\theta l_i : 0 \leqslant i \leqslant n-1)) = (1-|\theta(\lambda_n)|^2)\, \Pi_n(\lambda_n),$$

where the functions $\{\theta_i\}_{i=1}^n$ are defined by the Schur–Nevanlinna procedure (0.2).

Proof. Denote $d_n = \operatorname{dist}_{H^2}(P_\theta l_n, \operatorname{span}\{P_\theta l_i\}_{i=0}^{n-1})$. By Proposition A,

$$d_n^2 = \frac{\det \Gamma(\{P_\theta l_i\}_{i=0}^n)}{\det \Gamma(\{P_\theta l_i\}_{i=0}^{n-1})}.$$

Applying Proposition 2.1, identity (2.7), and definition (0.2) we obtain

$$
\begin{aligned}
d_n^2 &= (1-|\gamma_n|^2) \prod_{i=0}^{n-1} \frac{|1-\overline{\gamma}_i \theta_i(\lambda_n)|^2}{1-|\gamma_i|^2} \\
&= (1-|\gamma_n|^2) \prod_{i=0}^{n-1} \frac{1-|\theta_i(\lambda_n)|^2}{1-(|\theta_i(\lambda_n)-\gamma_i|^2/|1-\overline{\gamma}_i\theta_i(\lambda_n)|^2)} \\
&= (1-|\theta(\lambda_n)|^2)\, \Pi_n(\lambda_n). \quad \blacksquare
\end{aligned} \tag{3.1}
$$

PROPOSITION 3.2.

$$
\begin{aligned}
&\operatorname{dist}^2_{H^2}(k_\theta(\lambda_k,\cdot), \operatorname{span}(k_\theta(\lambda_i,\cdot): 0 \leqslant i \leqslant n,\, i \neq k)) \\
&\qquad = \left|\frac{B_n}{b_{\lambda_k}}(\lambda_k)\right|^2 \frac{1-|\theta(\lambda_k)|^2}{1-|\lambda_k|^2}\, \Pi_n(\lambda_k),
\end{aligned}
$$

where the functions $\{\theta_i\}_{i=1}^n$ *are defined by the Schur–Nevanlinna procedure* (0.2) *with "nodes"* $\lambda_0, \lambda_1, ..., \lambda_{k-1}, \lambda_{k+1}, ..., \lambda_n, \lambda_k$.

This proposition is proved similarly to the previous one; we use Corollary 2.9 instead of Proposition 2.1.

In the next two corollaries we deal with infinite sequences of distinct points $\Lambda = \{\lambda_n\}_{n=0}^\infty$ satisfying the Blaschke condition. Proposition 3.2 implies immediately the following criterion of minimality of $\mathscr{K}_{\theta,\Lambda}$.

COROLLARY 3.3. *The following conditions are equivalent*:

(a) *The family* $\{k_\theta(\lambda_i,\cdot)\}_{i=0}^\infty$ *is minimal.*

(b) *For every* κ *we have* $\Pi_\infty(\lambda_k) > 0$, *where the functions* $\{\theta_i\}_{i=1}^\infty$ *are defined by the Schur–Nevanlinna procedure* (0.2) *with "nodes"* $\lambda_0, \lambda_1, ..., \lambda_{k-1}, \lambda_{k+1}, ...$

In the next section we show that the convergence of the product $\Pi_\infty(\lambda_k)$ characterizes a geometric property of the class $\theta\overline{B} + H^\infty$ in L^∞/H^∞.

Let us turn to the uniform minimality.

COROLLARY 3.4. *The family* $\{k_\theta(\lambda_i,\cdot)\}_{i=0}^\infty$ *is uniformly minimal if and only if the following two conditions are fulfilled.*

(a) *The set* Λ *satisfies the Carleson condition* (C).

(b) $\Pi_\infty(\lambda_k) \geqslant C > 0$, *where the functions* $\{\theta_i\}_{i=1}^\infty$ *are defined by the Schur–Nevanlinna procedure* (0.2) *with "nodes"* $\lambda_0, ..., \lambda_{k-1}, \lambda_{k+1}, ...,$ *and the constant* C *does not depend on* k.

Proof. It is easily seen that $\|k_\theta(\lambda, \cdot)\|^2 = (1-|\theta(\lambda)|^2)/(1-|\lambda|^2)$. Then by Proposition 3.2,

$$\operatorname{dist}^2_{H^2}\left(\frac{k_\theta(\lambda_k, \cdot)}{\|k_\theta(\lambda_k, \cdot)\|}, \operatorname{span}(k_\theta(\lambda_k, \cdot) \colon 0 \leqslant i \leqslant n, i \neq k\right) = \left|\frac{B_n}{b_{\lambda_k}}(\lambda_k)\right|^2 \Pi_n(\lambda_k). \tag{3.2}$$

It remains only to note that $\Pi_n(\lambda_k) \leqslant 1$ and that $|(B_n/b_{\lambda_k})(\lambda_k)| \leqslant 1$. ▮

In particular, formula (3.2) means that the angles between the reproducing kernels decrease under the action of the projection P_θ.

The fact that the uniform minimality of the family $\mathcal{K}_{\theta, \Lambda}$ implies the Carleson condition on the sequence Λ was earlier established under the additional assumption $\sup_k |\theta(\lambda_k)| < 1$, see [HNP; N, Lecture 8].

In the following corollaries we do not assume $\lambda_0, \lambda_1, ..., \lambda_n, ...$ to be distinct.

Corollary 3.5. *Let $\mu \in \mathbb{D}$. Put $a_n(\mu) = (1-|\theta(\mu)|^2)\, \Pi_n(\mu)$, $n = 1, 2, ...$, where the numbers γ_n and the functions θ_n are defined by θ and Λ in the Schur–Nevanlinna procedure* (0.2). *Then the sequence $\{a_n(\mu)\}_{n=1}^\infty$ converges and*

$$\lim_{n\to\infty} a_n(\mu) = \operatorname{dist}^2_{H^2}\left(P_\theta\left(B\,\frac{(1-|\mu|^2)^{1/2}}{1-\bar{\mu}z}\right), P_\theta K_B\right). \tag{3.3}$$

Proof. By Proposition 3.1 we have

$$a_n(\mu) = \operatorname{dist}^2_{H^2}\left(P_\theta B_{n-1}\frac{\sqrt{1-|\mu|^2}}{1-\bar{\mu}z}, P_\theta K_{B_{n-1}}\right).$$

Taking into account that $B_n \to B$ in H^2, $P_\theta K_{B_n} \subset P_\theta K_{B_{n+1}}$, $n \geqslant 1$, and, consequently, $\operatorname{clos} \bigcup_{n=1}^\infty P_\theta K_{B_n} = P_\theta K_B$, and using an elementary Hilbert space geometry argument we get (3.3). ▮

Analogously (using Proposition 3.2 instead of Proposition 3.1) we obtain

Corollary 3.6. *In the conditions of Corollary* 3.5 *we have*

$$\frac{|B(\mu)|^2}{1-|\mu|^2}\lim_{n\to\infty} a_n(\mu) = \operatorname{dist}^2_{H^2}(k_\theta(\mu, \cdot), P_\theta K_B).$$

4. MINIMAL FAMILIES AND EXTREME POINTS

Suppose that θ is an inner function, B is a Blaschke product (not necessarily with simple zeros), and $\Lambda = \{\lambda_0, \lambda_1, ..., \lambda_n, ...\}$ is its zero sequence. Recall that $a_n(\mu) = (1-|\theta(\mu)|^2) \prod_{i=1}^n (1-|\theta_i(\mu)|^2)/(1-|\theta_i(\mu)|^2 |b_{\lambda_{i-1}}(\mu)|^2)$.

THEOREM 4.1. *In the conditions of Corollary* 3.5,

$$\lim_{n\to\infty} a_n(\mu) = \max_{f \in (\theta + b_\mu BH^\infty) \cap \mathscr{B}} \exp \int_0^{2\pi} \log(1-|f(e^{i\varphi})|^2)\, P(\varphi; \mu)\, dm(\varphi), \tag{4.1}$$

where $P(\varphi; \mu)$ is the Poisson kernel, $\mathscr{B} = \{f \in H^\infty : \|f\|_\infty \leqslant 1\}$.

Theorem 4.1 is a corollary of the following finite-dimensional result.

LEMMA 4.2. *For every $\mu \in \mathbb{D}$,*

$$a_n(\mu) = \max_{f \in (\theta + b_\mu B_{n-1} H^\infty) \cap \mathscr{B}} \exp \int_0^{2\pi} \log(1-|f(e^{i\varphi})|^2)\, P(\varphi; \mu)\, dm(\varphi),$$

and the maximum in the right-hand side is attained at a unique point F_n. In particular, the sequence $\{a_n(\mu)\}_{n=1}^\infty$ does not increase.

To prove Lemma 4.2 we need the following pointwise estimates on the unit circle $\mathbb{T}$.

LEMMA 4.3. (a) *For every $n \geqslant 0$ and $f \in (\theta + B_n H^\infty) \cap \mathscr{B}$,*

$$1-|f(z)|^2 \leqslant \prod_{k=0}^n \frac{|1-\overline{\gamma_k} f_k(z)|^2}{1-|\gamma_k|^2}, \qquad a.e. \quad z \in \mathbb{T}, \tag{4.2}$$

where $\{f_k\}_{k=0}^n$ is the sequence of the Schur–Nevanlinna functions corresponding to the function f and the sequence $\{\lambda_k\}_{k=0}^n$.

(b) *For every $n \geqslant 0$ there exists a unique function $F_n \in (\theta + B_n H^\infty) \cap \mathscr{B}$ such that*

$$1-|F_n(z)|^2 = (1-|\gamma_n|^2) \prod_{k=0}^{n-1} \frac{|1-\overline{\gamma_k} (F_n)_k(z)|^2}{1-|\gamma_k|^2}, \qquad a.e. \quad z \in \mathbb{T}, \tag{4.3}$$

where $\{(F_n)_k\}_{k=0}^{n-1}$ is the sequence of the Schur–Nevanlinna functions corresponding to the function F_n and the sequence $\{\lambda_k\}_{k=0}^{n-1}$.

Set $\tau_k(z) = (z-\lambda_k)/(1-\overline{\lambda_k} z)$. As in Section 2, put $B_n(z) = B_{0,n}(z) = \prod_{l=0}^n b_{\lambda_l}(z)$, $B_{k,n}(z) = \prod_{l=k}^n b_{\lambda_l}(z)$, $1 \leqslant k \leqslant n$, and $B_{n+1,n}(z) \equiv 1$. We need the following simple fact from [B].

LEMMA 4.4. *Let $f \in (\theta + B_n H^\infty) \cap \mathscr{B}$ and $0 \leqslant k \leqslant n$. Then*

$$f_k \in \theta_k + B_{k,n} H^\infty, \qquad 0 \leqslant k \leqslant n+1,$$

where $\{f_k\}_{k=0}^n$ is the sequence of the Schur–Nevanlinna functions corresponding to the function f and the sequence $\{\lambda_k\}_{k=0}^n$.

Proof of Lemma 4.3. (a) We use the notation $\gamma_k = \gamma_k(\theta; \{\lambda_l\}_{l=0}^k)$. Let $f \in \theta + B_n H^\infty$. Then by (0.2) and Lemma 4.4,

$$\gamma_k(f; \{\lambda_l\}_{l=0}^k) = f_k(\lambda_k) = \theta_k(\lambda_k) = \gamma_k, \qquad 0 \leqslant k \leqslant n.$$

Therefore,

$$f_{k+1}(z) = \frac{f_k(z) - \gamma_k}{1 - \overline{\gamma_k} f_k(z)} \frac{1}{\tau_k(z)}, \qquad 0 \leqslant k \leqslant n, \tag{4.4}$$

and as a consequence,

$$f_k(z) = \frac{f_{k+1}(z)\, \tau_k(z) + \gamma_k}{1 + \overline{\gamma_k} f_{k+1}(z)\, \tau_k(z)}, \qquad 0 \leqslant k \leqslant n. \tag{4.5}$$

Taking into account (2.7) we obtain that for a.e. $z \in \mathbb{T}$

$$1 - |f_k(z)|^2 = \frac{(1 - |\gamma_k|^2)(1 - |f_{k+1}(z)|^2)}{|1 + \overline{\gamma_k} f_{k+1}(z)\, \tau_k(z)|^2}, \qquad 0 \leqslant k \leqslant n.$$

As a result,

$$1 - |f(z)|^2 = (1 - |f_{n+1}(z)|^2) \prod_{k=0}^{n} \frac{(1 - |\gamma_k|^2)}{|1 + \overline{\gamma_k} \tau_k(z)\, f_{k+1}(z)|^2}, \tag{4.6}$$

where $f_{n+1} \in (\theta_{n+1} + B_{n,n+1} H^\infty) \cap \mathscr{B} = (\theta_{n+1} + H^\infty) \cap \mathscr{B}$, and consequently,

$$1 - |f(z)|^2 \leqslant \prod_{k=0}^{n} \frac{(1 - |\gamma_k|^2)}{|1 + \overline{\gamma_k} \tau_k(z)\, f_{k+1}(z)|^2}. \tag{4.7}$$

Substituting (4.4) in (4.7) we obtain (4.2).

(b) It follows from (4.6) that (4.7) turns into equality for $f = F_n$ if and only if $(F_n)_{n+1} \equiv 0$, that is, $(F_n)_n$ identically equals to γ_n. Now, formula (4.5) shows that the functions $(F_n)_k$, $0 \leqslant k \leqslant n$, and, in particular, the function F_n itself are uniquely determined by this condition. ∎

Proof of Lemma 4.2. We use Lemma 4.3 with λ_n replaced by μ. Applying the Poisson formula to (4.2) and (4.3) and using that, by Lemma 4.4, $(F_n)_k(\mu) = f_k(\mu) = \theta_k(\mu)$, $0 \leqslant k \leqslant n$, we obtain that

$$\exp \int_0^{2\pi} \log(1-|f(e^{i\varphi})|^2)\, P(\varphi;\mu)\, dm(\varphi)$$
$$\leqslant (1-|\theta_n(\mu)|^2) \prod_{k=0}^{n-1} \frac{|1-\overline{\gamma_k}\theta_k(\mu)|^2}{1-|\gamma_k|^2},$$
$$\exp \int_0^{2\pi} \log(1-|F_n(e^{i\varphi})|^2)\, P(\varphi;\mu)\, dm(\varphi)$$
$$= (1-|\theta_n(\mu)|^2) \prod_{k=0}^{n-1} \frac{|1-\gamma_k\theta_k(\mu)|^2}{1-|\gamma_k|^2}. \tag{4.8}$$

Formula (3.1) permits us to derive the assertion of the lemma from (4.8). ∎

Proof of Theorem 4.1. Let $F_n \in (\theta + b_\mu B_{n-1} H^\infty) \cap \mathscr{B}$, $n = 0, 1, ...$ be the functions from Lemma 4.2. By the Banach–Alaoglu theorem there exists a subsequence of the sequence $\{F_n\}_{n=0}^\infty$ such that $\{F_n\}_{n=0}^\infty$ converge to $F \in \mathscr{B}$, and $\{|F_n|^2\}_{n=0}^\infty$ converge to $G \in L^\infty$, $\|G\| \leqslant 1$ in the weak* topology of L^∞. Since the weak* convergency in $\mathscr{B}$ implies the uniform convergency on compact subsets of the unit disc, we have $F \in \theta + b_\mu BH^\infty$. By Lemma 4.2

$$\exp \int_0^{2\pi} \log(1-|F(e^{i\varphi})|^2)\, P(\varphi;\mu)\, dm(\varphi) \leqslant \lim_{n\to\infty} a_n(\mu).$$

Let us prove the opposite inequality. Clearly, $G(z) \geqslant |F^2(z)|$, a.e. $z \in \mathbb{T}$. Indeed, for every measurable $e \subset \mathbb{T}$,

$$2\int_e |F(e^{i\varphi})|^2\, dm(\varphi) = 2\int_e F(e^{i\varphi})\, \overline{F(e^{i\varphi})}\, dm(\varphi)$$
$$= 2 \lim_{n\to\infty} \int_e F_n(e^{i\varphi})\, \overline{F(e^{i\varphi})}\, dm(\varphi)$$
$$\leqslant \int_e |F(e^{i\varphi})|^2\, dm(\varphi) + \limsup_{n\to\infty} \int_e |F_n(e^{i\varphi})|^2\, dm(\varphi),$$

that is

$$\int_e |F(e^{i\varphi})|^2\, dm(\varphi) \leqslant \int_e G(e^{i\varphi})\, dm(\varphi).$$

Therefore, it is sufficient to verify that

$$\limsup_{n\to\infty} \exp \int_0^{2\pi} \log(1-|F_n(e^{i\varphi})|^2)\, dm(\varphi) \leqslant \exp \int_0^{2\pi} \log(1-G(e^{i\varphi}))\, dm(\varphi).$$

Put $e_\varepsilon=\{\varphi\in[0;2\pi]: 1-G(e^{i\varphi})>\varepsilon\}$, $\varepsilon>0$, $e_0=\bigcup_{\varepsilon>0} e_\varepsilon$, $e=\mathbb{T}\backslash e_0$, $b_n(e^{i\varphi})=(1-|F_n(e^{i\varphi})|^2)/(1-G(e^{i\varphi}))$.

If $m(e_\varepsilon)>0$, then $\{b_n\}_{n=0}^\infty$ converges to 1 in the weak* topology of $L^\infty(e_\varepsilon)$, and using the Jensen inequality

$$\exp\frac{1}{m(e_\varepsilon)}\int_{e_\varepsilon}\log g(e^{i\varphi})\,dm(\varphi)\leqslant\frac{1}{m(e_\varepsilon)}\int_{e_\varepsilon} g(e^{i\varphi})\,dm(\varphi),\qquad g\geqslant 0, \tag{4.9}$$

we get

$$\begin{aligned}\limsup_{n\to\infty}\ &\frac{1}{m(e_\varepsilon)}\int_{e_\varepsilon}\log(1-|F_n(e^{i\varphi})|^2)\,dm(\varphi)\\ &-\frac{1}{m(e_\varepsilon)}\int_{e_\varepsilon}\log(1-|G(e^{i\varphi})|)\,dm(\varphi)\\ &\leqslant\limsup_{n\to\infty}\frac{1}{m(e_\varepsilon)}\int_{e_\varepsilon}\log b_n(e^{i\varphi})\,dm(\varphi)\\ &\leqslant\limsup_{n\to\infty}\log\left[\frac{1}{m(e_\varepsilon)}\int_{e_\varepsilon} b_n(e^{i\varphi})\,dm(\varphi)\right]=0.\end{aligned}$$

Since $\log(1-|F_n(e^{i\varphi})|^2)\leqslant 0$ and $\log(1-|G(e^{i\varphi})|^2)\leqslant 0$, we have

$$\limsup_{n\to\infty}\int_{\mathbb{T}}\log(1-|F_n(e^{i\varphi})|^2)\,dm(\varphi)\leqslant\int_{e_0}\log(1-|G(e^{i\varphi})|)\,dm(\varphi).$$

Finally, if $m(e)>0$, then we use (4.9) again and obtain (since $|F_n|^2$ converge to 1 in the weak* topology of $L^\infty(e)$) that

$$\begin{aligned}\limsup_{n\to\infty}&\frac{1}{m(e)}\int_e\log(1-|F_n(e^{i\varphi})|^2)\,dm(\varphi)\\ &\leqslant\limsup_{n\to\infty}\log\left[\frac{1}{m(e)}\int_e(1-|F_n(e^{i\varphi})|^2)\,dm(\varphi)\right]=-\infty,\end{aligned}$$

and, consequently,

$$\limsup_{n\to\infty}\int_{\mathbb{T}}\log(1-|F_n(e^{i\varphi})|^2)\,dm(\varphi)=-\infty. \quad\blacksquare$$

We use an evident corollary of Proposition F:

LEMMA 4.5. *Let θ, B be inner functions. Then the following two assertions are equivalent*:

(a) $\theta\overline{B}+H^\infty$ *is an extreme point of the closed unit ball in* L^∞/H^∞.

(b) $(\theta+BH^\infty)\cap\mathscr{B}=\{\theta\}$.

Remark. It follows from a result in [ØS] that for every inner function θ there exists an interpolating Blaschke product B that satisfies (together with θ) the conditions of Lemma 4.5.

Theorem 4.1 permits us to obtain a criterion for a class $\theta\overline{B}+H^\infty$ in L^∞/H^∞ to be an extreme point of the closed unit ball.

COROLLARY 4.6. *The following assertions are equivalent*:

(a) *For some* $\mu\in\mathbb{D}$ *we have* $\lim_{n\to\infty} a_n(\mu)=0$.

(b) *For every* $\mu\in\mathbb{D}$ *we have* $\lim_{n\to\infty} a_n(\mu)=0$.

(c) *For some* $\mu\in\mathbb{D}$ *the class* $\theta\overline{b_\mu B}+H^\infty$ *is an extreme point of the closed unit ball in* L^∞/H^∞.

(d) *For every* $\mu\in\mathbb{D}$ *the class* $\theta\overline{b_\mu B}+H^\infty$ *is an extreme point of the closed unit ball in* L^∞/H^∞.

In the proof of the corollary we use that the integrals in the right-hand side of (4.1) vanish for all $\mu\in\mathbb{D}$ simultaneously.

Note that if one (all) of the assertions (a)–(d) of Corollary 4.6 is (are) fulfilled, then the function F maximizing the left-hand side of (4.1) coincides with θ.

THEOREM 4.7. *Let* θ *be an inner function and let* B *be the Blaschke product constructed by a sequence* Λ *with distinct points. Then the following assertions are equivalent*:

(a) *The family* $\mathscr{K}_{\Lambda,\theta}$ *is not minimal.*

(b) *For some (every)* $\mu\in\Lambda$ *the family* $\mathscr{K}_{\Lambda\setminus\{\mu\},\theta}$ *is complete in* K_θ.

(c) *For some (every)* $\mu\in\Lambda$, $P_\theta(B/(z-\mu))\in\operatorname{clos} P_\theta(K_{B/b_\mu})$.

(d) *For some (every)* $\mu\in\Lambda$, $P_\theta(B/(z-\mu)\,H^2)\subset\operatorname{clos} P_\theta(K_{B/b_\mu})$.

(e) *The class* $\theta\overline{B}+H^\infty$ *is an extreme point of the closed unit ball of* L^∞/H^∞.

(f) $(\theta+BH^\infty)\cap\mathscr{B}=\{\theta\}$.

Proof. Denote by (be), (ce), (de) the "every" versions of assertions (b)–(d), and by (bs), (cs), (ds) the "some" ones. Then we have (e) $\Leftrightarrow$ (f) by Lemma 4.5, (cs) $\Rightarrow$ (e) by Corollaries 3.5 and 4.6, (e) $\Rightarrow$ (a) by Corollaries 3.6 and 4.6, (a) $\Rightarrow$ (be) by Proposition E. The implications (bs) $\Rightarrow$ (ds) $\Rightarrow$ (cs), (be) $\Rightarrow$ (de) $\Rightarrow$ (ce), (be) $\Rightarrow$ (bs), and (ce) $\Rightarrow$ (cs) are evident. ∎

Remark. Certainly, (de) of Theorem 4.7 implies that $P_\theta B \in \operatorname{clos} P_\theta(K_B)$. However, the converse implication does not hold (e.g., in the case $\theta = B$).

Proposition C shows that if $\mathcal{K}_\Lambda$ is uniformly minimal, then it is an unconditional basis in the closure of its linear hull. In a special case we obtain an analog of this property concerning families $\mathcal{K}_{\Lambda,\theta}$.

COROLLARY 4.8. *Let Λ satisfy the Carleson condition* (C) *and let* $\lim_{n\to\infty} \theta(\lambda_n) = 0$. *Then the following assertions are equivalent*:

(a) *The family $\mathcal{K}_{\Lambda,\theta}$ is minimal.*

(b) *The family $\mathcal{K}_{\Lambda,\theta}$ is an unconditional basis in the closure of its linear hull.*

(c) *The class $\theta + BH^\infty$ contains a non-extreme point of $\mathcal{B}$.*

(d) $\operatorname{dist}_{L^\infty}(\theta, BH^\infty) < 1$.

Proof. The implications (b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c) are evident. Proposition D implies that (b) $\Leftrightarrow$ (d). Theorem 4.7 gives the equivalence (a) $\Leftrightarrow$ (c). It remains to verify the implication (a) $\Rightarrow$ (d). Suppose that $\mathcal{K}_{\Lambda,\theta}$ is minimal. Then, by Theorem 4.7, the class $\theta + BH^\infty \cap \mathcal{B}$ contains at least two functions. On the other hand, Proposition G implies that this class contains a unique element of minimal norm. Thus, the norm of the class is less than 1. ∎

Clearly, Corollary 4.8 is of interest only in the case where θ is a Blaschke product. Otherwise, Proposition G (see also [HNP, Part III]) ensures the inequality (0.1).

The conditions of Corollary 4.8 do not imply inequality (0.1). It is easy to show that there exist Blaschke products $F(z)$ and $G(z) = \prod_{i=0}^\infty b_{\mu_i}(z)$ such that $\lim_{i\to\infty} F(\mu_i) = 0$ and $\{\mu_i\}_{i=0}^\infty$ satisfies the Carleson condition (C) while the inequality (0.1) is not fulfilled. We can just take $B = z\theta$, where θ is an arbitrary Blaschke product satisfying the condition (C).

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